Two-step reliability test based unitary root-MUSIC for direction-of-arrival estimation ☆

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ABSTRACT

A two-step reliability test (TSRT) based unitary root-MUSIC algorithm for direction-of-arrival (DOA) estimation is proposed in this paper. We combine the conventional beamforming and unitary root-MUSIC to compute the DOA estimates and employ the pseudo-noise resampling (PR) technique to construct a DOA estimator bank. Unlike the standard reliability test, we devise the TSRT which retains the successful DOA estimates of a given DOA estimator separately to construct a DOA estimate set that is used to determine the final DOA estimates. Compared to the existing PR based DOA estimation methods, our solution can achieve better threshold performance by using fewer PR runs. Furthermore, the TSRT can be easily applied to other DOA estimation methods. Simulations verify the effectiveness of the proposed scheme.

1. Introduction

Direction-of-arrival (DOA) estimation of signals impinging upon a sensor array is a fundamental problem in many fields such as radar, sonar and wireless communications [1–6]. Among them, the maximum likelihood (ML) [1] and subspace based methods [2–6] have been studied extensively. The ML method is theoretically optimal and it is equivalent to the least squares estimator under the assumption of white Gaussian noise. Although the ML estimate has excellent statistical properties, it requires a nonlinear and multidimensional optimization procedure which is computationally intensive. In contrast, the subspace based algorithms offer a good compromise between resolution and computational complexity. At high signal-to-noise ratio (SNR) and large sample regimes, they can offer comparable performance to the ML method, but with a much lower computational cost. All estimators for nonlinear parameters suffer threshold effect [7–9]. But some will have threshold effect at higher SNR, and some at lower SNR. The most visual embodiment of this phenomenon is that the observed estimation errors rapidly depart from the Cramér–Rao lower bound (CRLB) when either the number of samples or SNR is below a certain threshold.

To circumvent this issue, a vast number of techniques have been proposed in the literature. Gershman [10] has developed a joint estimation strategy, which selects the final DOA estimates from a pre-generated estimator bank consisting of several DOA estimators calculated simultaneously from a set of data or a given sample covariance matrix to reduce the performance degradation caused by outliers. In [11], a pseudo-random spatial spectrum resampling technique has been used to improve the threshold performance of subspace based DOA estimation methods. The essence of this approach is to resample the signal or noise subspace using a weighted MUSIC method with randomly generated weighting matrices. Although this technique provides a considerable threshold improvement for the MUSIC algorithm, it seems to be too specific in that it cannot be applied in the same way to a variety of existing DOA estimation techniques. Motivated by the success of modern resampling schemes (e.g., bootstrap [12,13]), pseudo-noise resampling (PR) technique has been presented in [14–18]. Its underlying idea is to utilize synthetically generated pseudo-noise to perturb the original noise in such a way that the outliers will be removed. In [15], a PR based unitary ESPRIT algorithm has been developed to mitigate the effect of complex eigenvalues which appear in pairs. Nevertheless, it cannot eliminate the effect caused by those DOA estimates deviated far away from the true DOAs. Vasylyshyn [16] has proposed a variant of the PR based root-MUSIC algorithm which combines the PR technique with the conventional

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Table 1
List of acronyms.

CB-URM	Conventional beamforming based unitary root-MUSIC
CRLB	Cramér–Rao lower bound
DDS	Distance detection strategy
DOA	Direction-of-arrival
ML	Maximum likelihood
PR	Pseudo-noise resampling
TSRT	Two-step reliability test
URM	Unitary root-MUSIC
ULA	Uniform linear array

beamforming based root-MUSIC. However, it essentially relies on the complex-valued computation which turns out to be computationally demanding, especially when the numbers of array elements and PR processes are large. Recently, we have proposed a PR based unitary root-MUSIC (PR-URM) algorithm [17]. The main contribution of [17] is the distance detection strategy (DDS), which is used to determine the final DOA estimates when all the DOA estimators fail to pass the reliability test. However, the DDS can only be applied to the root-MUSIC like algorithms since it needs the signal root information.

Although the previous PR based DOA estimation algorithms have achieved appealing effects, the performance improvement comes at the expense of a large number of PR runs. The existing PR based algorithms such as [14–18] have a common drawback that as long as there is an unsuccessful DOA estimate in the DOA estimator, the successful DOA estimates will be regarded as the unsuccessful one and they will be ruled out along with the whole estimator. Here, the successful DOA estimate denotes the DOA estimate localized within its DOA sector, while the unsuccessful DOA estimate is out of its DOA sector. This implies that the useful information of the successful DOA estimates will be neglected. More seriously, it may lead to the failure of the PR process. As a result, to achieve a favorable performance, these techniques need a vast number of PR processes, which may be not feasible in practice.

In this paper, a two-step reliability test (TSRT) based unitary root-MUSIC (URM) algorithm is devised. We only consider uniform linear array (ULA) in this study as URM is designed for it. With the result of [19], we first consider a conventional beamforming based unitary root-MUSIC (CB-URM) which is realized via combining the URM with conventional beamforming to obtain DOA estimator which contains P DOA estimates. Here, P is the number of sources. After PR processing, unlike the methods in [14–17], we propose the TSRT to help select successful DOA estimates. After dividing the DOA estimator bank into P DOA subsets, with each subset corresponding to a common DOA, the TSRTs between those subsets are statistically independent, which allows us to use fewer PR runs to collect enough successful DOA estimates that can be used to determine the final DOA estimates. Furthermore, the TSRT can be employed for other existing DOA estimation algorithms.

The remainder of the paper is organized as follows. The data model is presented in Section 2. The motivation and derivation of the proposed algorithm are provided in Section 3. Simulation results are given in Section 4. Finally, conclusions are drawn in Section 5. Acronyms used in the paper are given in Table 1.

2. Problem formulation

2.1. Signal model

Consider a ULA with M isotropic sensors. There are P ($P < M$) uncorrelated narrowband source signals impinging on the array from distinct directions $\{\theta_1, \cdots, \theta_P\}$ in the far field. The $M \times 1$ observation vector is

$$\mathbf{x}(t) = \mathbf{A}\mathbf{s}(t) + \mathbf{n}(t), \quad t = 1, \cdots, N. \tag{1}$$

Here, $\mathbf{A} = [\mathbf{a}(\theta_1), \cdots, \mathbf{a}(\theta_P)]$ is the steering matrix, $\mathbf{s}(t) = [s_1(t), \cdots, s_P(t)]^T$ is the source signal vector with $(\cdot)^T$ being the transpose, N is the number of snapshots and the steering vector due to the pth source is expressed as

$$\mathbf{a}(\theta_p) = [1, e^{j2\pi \sin(\theta_p)d/\lambda}, \cdots, e^{j2\pi (M-1)\sin(\theta_p)d/\lambda}]^T \tag{2}$$

where λ is the carrier wavelength and $d = \lambda/2$ is the inter-element spacing. It is assumed that the noise vector $\mathbf{n}(t)$ is a white Gaussian process with mean zero and covariance $\sigma_n^2\mathbf{I}_M$ where σ_n^2 is the power and $\mathbf{I}_M$ is the $M \times M$ identity matrix. Moreover, the noise is uncorrelated with $\mathbf{s}(t)$. Our task is to estimate the P DOAs from the observed vector $\mathbf{x}(t)$. The covariance matrix of $\mathbf{x}(t)$ is

$$\mathbf{R} = \mathbb{E}[\mathbf{x}(t)\mathbf{x}(t)^H] = \mathbf{A}\mathbf{R}_s\mathbf{A}^H + \sigma_n^2\mathbf{I}_M \tag{3}$$

where $\mathbb{E}[\cdot]$ is the mathematical expectation, $(\cdot)^H$ is the conjugate transpose and $\mathbf{R_s} = \mathbb{E}[\mathbf{s}(t)\mathbf{s}(t)^H]$ denotes the signal covariance matrix.

2.2. Unitary root-MUSIC algorithm

The URM algorithm [6] utilizes a real-valued covariance matrix given as

$$\mathbf{C} = \frac{1}{2}\mathbf{Q}_M^H \left(\mathbf{R} + \mathbf{J}_M\mathbf{R}^*\mathbf{J}_M\right)\mathbf{Q}_M$$
$$= \mathrm{Re}\left\{\mathbf{Q}_M^H\mathbf{R}\mathbf{Q}_M\right\} \tag{4}$$

where $\mathrm{Re}\{\cdot\}$ represents the real part, $\mathbf{J}_M$ is an $M \times M$ exchange matrix with ones on its anti-diagonal and zeros elsewhere, $(\cdot)^*$ represents complex conjugate and $\mathbf{Q}_M$ is a sparse unitary matrix, defined as [6,15]

$$\mathbf{Q}_M = \begin{cases} \dfrac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_l & j\mathbf{I}_l \\ \mathbf{J}_l & -j\mathbf{J}_l \end{bmatrix}, & \text{for } M = 2l \\[12pt] \dfrac{1}{\sqrt{2}} \begin{bmatrix} \mathbf{I}_l & \mathbf{0}_l & j\mathbf{I}_l \\ \mathbf{0}_l^T & \sqrt{2} & \mathbf{0}_l^T \\ \mathbf{J}_l & \mathbf{0}_l & -j\mathbf{J}_l \end{bmatrix}, & \text{for } M = 2l + 1. \end{cases} \tag{5}$$

Here, $\mathbf{0}_l$ is an $l \times 1$ zero vector. Define the eigenvalue decomposition of $\mathbf{C}$ as

$$\mathbf{C} = \mathbf{E}\mathbf{\Lambda}\mathbf{E}^H = \mathbf{E}_S\mathbf{\Lambda}_S\mathbf{E}_S^H + \sigma_n^2\mathbf{E}_N\mathbf{E}_N^H \tag{6}$$

where $\mathbf{E}_S = [\mathbf{e}_1, \cdots, \mathbf{e}_P]$, $\mathbf{\Lambda}_S = \mathrm{diag}\{\lambda_1, \cdots, \lambda_P\}$ and $\mathbf{E}_N = [\mathbf{e}_{P+1}, \cdots, \mathbf{e}_M]$ with $\{\lambda_i\}_{i=1}^P$ being the signal eigenvalues, $\{\mathbf{e}_i\}_{i=1}^P$ being its corresponding signal eigenvectors, $\{\mathbf{e}_i\}_{i=P+1}^M$ being the noise eigenvectors and $\mathrm{diag}\{\cdot\}$ being a diagonal matrix. Then the unitary root-MUSIC polynomial can be expressed as

$$f_{\text{U-MUSIC}}(z) = \tilde{\mathbf{a}}^T(1/z)\mathbf{E}_N\mathbf{E}_N^T\tilde{\mathbf{a}}(z) \tag{7}$$

where $\tilde{\mathbf{a}}(z) = \mathbf{Q}_M^H\mathbf{a}(z)$ with $z_i = e^{j2\pi d \sin\theta_i/\lambda}$ being the root of (7). Through finding the P roots which are closest to the unit circle, we determine the DOAs:

$$\theta_i = \sin^{-1}\left(\frac{\angle(z_i)\lambda}{2\pi d}\right), \quad i = 1, \cdots, P \tag{8}$$

where $\angle$ represents the angle operator.

3. Proposed algorithm

It has been shown in [6] that the URM has a better performance than that of the root-MUSIC but with a much lower computational complexity since it is realized in terms of real-valued computation. However, it will suffer performance degradation, especially in the low SNR and small sample scenarios. This is due to the fact that the URM estimator cannot efficiently handle the outliers. To circumvent this issue, a TSRT based URM approach is devised for computationally efficient and accurate DOA estimation.
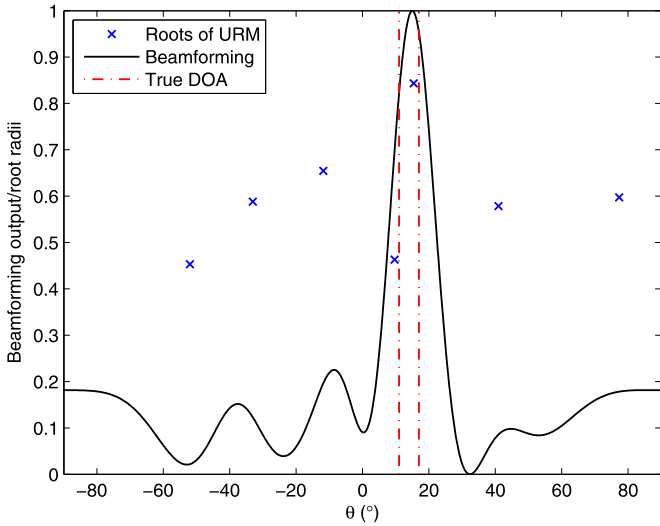
Fig. 1. Conventional beamforming output and root radii of URM. $M = 8$, $N = 10$, SNR $= -7$ dB and two equal-power sources imping from $11°$ and $18°$.

3.1. Conventional beamforming based URM

In this section, motivated by [19], the CB-URM algorithm is considered. Unlike the URM which determines the DOA estimates by choosing the roots that are closest to the unit circle, the CB-URM identifies the true signal roots by using the maximum output of conventional beamforming towards the previously estimated DOA candidates. Fig. 1 shows the conventional beamforming output and root radii of URM. According to the URM, the third and fifth roots at $-11.8°$ and $15.4°$, respectively, whose magnitudes are closest to one, are chosen as the signal roots. However, the fourth root at $9.6°$ is reasonably a better choice than the third one. This is because, at low SNRs and small samples, the signal and noise roots may be merged together. In other words, the noise roots may have larger modulus than the signal roots. Thus, the standard way is not always able to correctly determine the signal roots. In contrast, the beamforming output depends on the response of the signal roots rather than the modulus. Therefore, using the beamforming output to determine the signal roots is preferred over the standard approach.

3.2. Pseudo-noise resampling process

The key of the proposed method is to test the following hypothesis $\mathcal{H}$ for each DOA estimator [15–18], which enables us to separate the normal and abnormal DOA estimators.

$\mathcal{H}$: All the DOA estimates in a DOA estimator are located in $\hat{\Theta}$.

Here, $\hat{\Theta}$ contains the pre-estimated angular sectors of source localization. A simple way of determining $\hat{\Theta}$ is to use the conventional beamforming [15–18]. Let

$$\hat{\Theta} = \hat{\Theta}_1 \cup \hat{\Theta}_2 \cup \cdots \cup \hat{\Theta}_F \tag{9}$$

where F is the number of peaks of beamforming output,

$$\hat{\Theta}_i = [\theta_i^{\max} - \theta_i^{\text{left}}, \theta_i^{\max} + \theta_i^{\text{right}}], \quad i = 1, \cdots, F$$

is the ith estimated DOA sector, $\theta_i^{\max}$ is the coordinate of the ith highest peak of the conventional beamforming output, θ_i^{left} and θ_i^{right} are the left and right boundaries of the ith subinterval where

both of them can be chosen as angular distances between the maximum of the ith peak and the left/right neighbor point with 3 dB drop, respectively. If the ith peak has no right or left 3 dB drop, the θ_i^{left} and θ_i^{right} can be chosen as the left and right boundary points of the ith peak's lobe. Note that when there are closely spaced DOAs or SNR is extremely small, the conventional beamforming might have only one peak. In this case, the number of estimated sectors will be smaller than the number of sources. Fortunately, it has almost no impact on the performance of the proposed method if all the true DOAs contained in $\hat{\Theta}$. This will be further demonstrated in Section 4.

When the hypothesis $\mathcal{H}$ is rejected, the data matrix $\mathbf{X} = [\mathbf{x}(1), \cdots, \mathbf{x}(N)]$ will be resampled K times using synthetically generated pseudo-noise. The $M \times N$ resampled data matrix is given as

$$\bar{\mathbf{X}} = \mathbf{X} + \mathbf{Y} \tag{10}$$

where $\mathbf{Y}$ is the $M \times N$ pseudo-noise matrix with mean zero and covariance matrix $\sigma_Y^2 \mathbf{I}_M$. Here, σ_Y^2 is the variance of the pseudo-noise and its value should be comparable with the variance of the measurement noise σ_n^2. It is shown in [15–17] that we can estimate σ_Y^2 as $\hat{\sigma}_Y^2 = p \cdot \hat{\sigma}_n^2$ where $p \approx 1$ is a user-defined parameter and $\hat{\sigma}_n^2$ is the consistent estimate of σ_n^2 given by

$$\hat{\sigma}_n^2 = \frac{1}{M - P} \sum_{i=P+1}^{M} \hat{\lambda}_i \tag{11}$$

with $\hat{\lambda}_1 \geq \cdots \geq \hat{\lambda}_M$ being the ordered eigenvalues of the sample covariance matrix $\hat{\mathbf{R}} = \mathbf{X}\mathbf{X}^H / N$.

3.3. Two-step reliability test

For each resampling run, we apply the CB-URM method to obtain a DOA estimator Υ which contains P DOA estimates. Let the ith estimator be

$$\Upsilon^{(i)} = \left[\hat{\theta}_1^{(i)}, \cdots, \hat{\theta}_P^{(i)} \right]^T \tag{12}$$

where $\hat{\theta}_1^{(i)} \leq \cdots \leq \hat{\theta}_P^{(i)}$ are the P DOA estimates obtained in the ith PR run. Assuming that after K PR runs, we have K estimators which are used to form a DOA estimator bank:

$$\begin{aligned}
\mathcal{B}_\theta &= \left[\Upsilon^{(1)}, \cdots, \Upsilon^{(K)} \right] \\
&= \begin{bmatrix}
\hat{\theta}_1^{(1)} & \hat{\theta}_1^{(2)} & \cdots & \hat{\theta}_1^{(K)} \\
\hat{\theta}_2^{(1)} & \hat{\theta}_2^{(2)} & \cdots & \hat{\theta}_2^{(K)} \\
\vdots & \vdots & \ddots & \vdots \\
\hat{\theta}_P^{(1)} & \hat{\theta}_P^{(2)} & \cdots & \hat{\theta}_P^{(K)}
\end{bmatrix}.
\end{aligned} \tag{13}$$

The existing methods [10–16] employ the reliability test $\mathcal{H}$ to divide $\mathcal{B}_\theta$ into two subsets: one subset contains J estimators that pass $\mathcal{H}$, and the other contains the remaining $(K - J)$ estimators for which this test fails. However, this approach has a main drawback that, for any tested DOA estimator, as long as there is an estimated DOA localized outside $\hat{\Theta}$, the whole estimator will be rejected by $\mathcal{H}$. In other words, those DOA estimates localized inside $\hat{\Theta}$ will also be rejected. At low SNRs and small samples, although the probability that the whole estimator accepted by $\mathcal{H}$ is low, the probability of a single DOA estimate being localized in $\hat{\Theta}$ is much higher. This phenomenon is supported by the following example. We use the URM [6] algorithm to form the estimator. Furthermore, we utilize the conventional beamforming to determine the estimator $\hat{\Theta}$. It is observed from Fig. 2 that when the SNR is smaller than 2 dB, the probabilities that θ_1 and θ_2 can be localized in $\hat{\Theta}$ separately are much higher than that of the whole estimator passing $\mathcal{H}$.

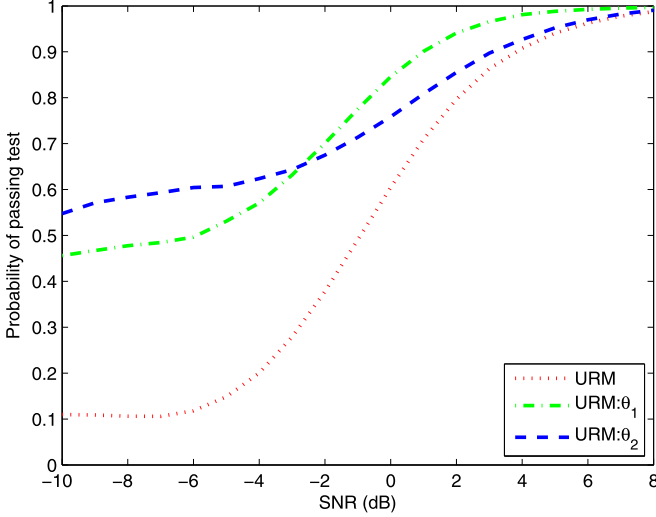**Fig. 2.** Probability of passing reliability test. $M = 8$, $N = 10$, $K = 10$ and two equal-power sources imping from $\theta_1 = 11°$ and $\theta_2 = 18°$.

Therefore, if we remove the DOA estimates localized out of $\hat{\Theta}$ (we call them "unsuccessful DOA estimates") and only retain those localized in $\hat{\Theta}$ (we call them "successful DOA estimates"), compared with the conventional method, it is possible to use fewer PR runs to collect enough DOA estimates that can be used to determine the final DOA estimates. Based on the analysis above, we now propose the TSRT method to divide the successful and unsuccessful DOA estimates. Recall that $\hat{\Theta}$ consists of P sub-sectors with each sub-sector corresponding to a true DOA, i.e., $\{\hat{\Theta}_i\}_{i=1}^{P}$. To proceed, we divide $\mathcal{B}_\theta$ into P subsets with each subset containing K DOA estimates that are corresponding to a common DOA and sorted in descending order. Let

$$\mathcal{S}_i = \left[\tilde{\theta}_i^{(1)}, \cdots, \tilde{\theta}_i^{(K)}\right] \tag{14}$$

be the ith subset with $\tilde{\theta}_i^{(1)} \leq \cdots \leq \tilde{\theta}_i^{(K)}$ being the sorted K DOA estimates in the ith row of $\mathcal{B}_\theta$. For the ith subset $\mathcal{S}_i$ ($i = 1, \cdots, P$), the TSRT utilizes the ith sub-sector $\hat{\Theta}_i$ to test K DOA estimates in $\mathcal{S}_i$. Assume that there are L_i DOA estimates localized in $\hat{\Theta}_i$. After completing K tests, we divide $\mathcal{S}_i$ into two parts:

$$\begin{cases} \mathcal{S}_{i,1} = \left[\tilde{\theta}_i^{(k_i+1)}, \cdots, \tilde{\theta}_i^{(k_i+L_i)}\right] & \text{(a)} \\ \mathcal{S}_{i,0} = \left[\tilde{\theta}_i^{(1)}, \cdots, \tilde{\theta}_i^{(k_i)}, \tilde{\theta}_i^{(k_i+L_i+1)}, \cdots, \tilde{\theta}_i^{(K)}\right] & \text{(b)} \end{cases} \tag{15}$$

where $\tilde{\theta}_i^{(k_i+1)}$ denotes the first DOA estimate that is localized in $\hat{\Theta}_i$, $\mathcal{S}_{i,1}$ contains L_i successful DOA estimates (i.e., "successful DOA subset") and $\mathcal{S}_{i,0}$ contains the remaining $(K - L_i)$ unsuccessful DOA estimates (i.e., "unsuccessful DOA subset").

Remark 1. The success of the TSRT is established when $\hat{\Theta}$ contains all the true DOAs. This is the prerequisite for a successful PR technique. All the PR based DOA estimation methods such as [15–18] need this prerequisite, otherwise the PR technique will be invalid. In fact, the TSRT can estimate all the DOAs successfully even if the number of sectors is smaller than the number of sources provided that all the sectors contains P true DOAs. For example, assume that there are three DOAs, namely, θ_1, θ_2 and θ_3. Consider the following cases: the θ_1 is widely spaced with the other two DOAs, and θ_2 and θ_3 are closely spaced that the beamforming cannot resolve them such that there are only two distinct peaks where one peak is at θ_1 and the other is around θ_2 and θ_3. According to (9), we obtain an estimate of $\hat{\Theta}$ that has two sectors. In this case, the number

of sectors in $\hat{\Theta}$ is smaller than 3. If $\hat{\Theta}$ contains the three DOAs, the proposed method is valid. However, if any of the three DOAs are localized out of $\hat{\Theta}$, the proposed method will be invalid, which is also true for all of the PR based DOA estimation methods. Moreover, for subspace based DOA estimation methods, accurate source number estimation is also a prerequisite for the success of these algorithms. If the number of sources is not correctly estimated, the proposed method will suffer severe performance degradation. The information theoretic criteria [21,22] or their variants [23–25] can be applied to estimate the number of sources.

3.4. Direction-of-arrival estimation

Although the probability of single DOA estimate localized in $\hat{\Theta}$ is much higher than that of the whole estimator, it is also observed from Fig. 2 that we cannot assume that there always exists P non-empty $\{\mathcal{S}_{i,1}\}_{i=1}^{P}$. In other words, for some subsets, all the DOA estimates may be localized out of their sectors. Consequently, we consider the following two cases:

1) The TSRT succeeds: each of the P subsets contains L_i ($L_i > 0$, $i = 1, \cdots, P$) successful DOA estimates.
2) The TSRT fails: there is at least one subset in which the K DOA estimates are localized out of their sectors.

For the first case, because $\mathcal{S}_{1,1}, \cdots, \mathcal{S}_{P,1}$, are all non-empty, we use the median to determine the final DOA estimates. The ith final DOA estimate can be determined by choosing the average of the two DOA estimates in the middle of the ith subset if K is even or choosing the median of the ith subset if K is odd, i.e.,

$$\hat{\theta}_i = \text{med}\left\{\tilde{\theta}_i^{(k_i+1)}, \cdots, \tilde{\theta}_i^{(k_i+L_i)}\right\}, \quad \text{for } i = 1, \cdots, P \tag{16}$$

where med$\{\cdot\}$ represents the median operator. More specifically, for arbitrary $b_1, \cdots, b_\ell$, we have

$$\text{med}\{b_1, \cdots, b_\ell\} = \begin{cases} (c_{\ell/2} + c_{1+\ell/2})/2, & \text{if } \ell \text{ is even} \\ c_{(\ell+1)/2}, & \text{if } \ell \text{ is odd} \end{cases} \tag{17}$$

where $\{c_1, \cdots, c_\ell\}$ represent the sorted $\{b_1, \cdots, b_\ell\}$.

For the second case, we assume that there are Q ($Q \leq P$) subsets containing no successful DOA estimates, while the remaining $(P - Q)$ subsets contains at least one successful DOA estimate. For the latter, we can directly apply (16) to determine their final DOA estimates. For the former, we use $\{q_1, \cdots, q_Q\}$ to represent the Q unsuccessful subsets. It is easy to find that for $\mathcal{S}_{q_i}(i = 1, \cdots, Q)$, $\mathcal{S}_{q_i,1}$ is empty and $\mathcal{S}_{q_i,0}$ contains K DOA estimates of $\mathcal{S}_{q_i}$. For this case, we can use the median or directly take the average of $\mathcal{S}_{q_i,0}$ to determine their final DOA estimates. The proposed method for DOA estimation is summarized in Table 2.

Remark 2. The main difference between the proposed and existing methods is the reliability test after obtaining the estimator bank. In our solution, the TSRT works on the P subsets separately, which implies that the test of each subset is independent of the others and the testing results of each subset are retained separately. This enables us to classify the successful DOA estimates as (15a). When $Q > 0$, the proposed approach uses the average or median to determine the Q final DOA estimates corresponding to Q unsuccessful subsets, while the remaining $(P - Q)$ DOAs can be determined via (15a) and (16). Meanwhile, in [17], we have shown that the performance of the DDS [17] is better than that of the median method when all the DOA estimates are rejected by $\mathcal{H}$. For the proposed algorithm, and considering the advantages of the DDS technique, it is much better to choose the DDS rather than the median to determine the final DOAs since there may still exist such a case in which there is at least one unsuccessful DOA subset.

Table 2
Pseudo-code of proposed method.

Step 1 Use an existing DOA estimation method to obtain P DOA estimates based on the data matrix **X**.
Step 2 Employ the conventional beamforming to pre-estimate the DOA sectors according to (9).
Step 3 Test the hypothesis $\mathcal{H}$ for this estimator.
• If $\mathcal{H}$ is accepted, terminate the algorithm.
• If $\mathcal{H}$ is rejected, utilize PR process to generate K resampled data matrices, and then apply the DOA estimation method in **Step 1** to form the estimator bank of (13).
Step 4 Apply the TSRT to each resampled estimator from the estimator bank.
CASE 1 If all the subsets have at least one successful DOA estimate, then estimate the ith DOA θ_i via (16).
CASE 2 • If there are Q $(0 < Q < P)$ subsets that are unsuccessfully localized in their sectors, then we employ the median or average to determine the Q final DOA estimates.
• For the remaining $(P - Q)$ subsets that are successfully localized in their sectors, estimate the remaining $(P - Q)$ DOA estimates via (16).

Remark 3. The complexity of the proposed method is mainly caused by constructing of the estimator bank $\mathcal{B}_\theta$. As we know, the CB-URM algorithm is employed K times to obtain $\mathcal{B}_\theta$. Thus, the complexity of the proposed method is approximately K times as that of the CB-URM. On the other hand, the implementation of the CB-URM mainly requires two steps: i) the estimation of **C**, i.e., $\hat{\mathbf{C}}$, which is about $\mathcal{O}(2M^2N)$; ii) the EVD of $\hat{\mathbf{C}}$ which is about $\mathcal{O}(M^3)$, resulting in an overall complexity of $\mathcal{O}(2M^2N + M^3)$. Therefore, the computational complexity of the proposed method is about $\mathcal{O}(K(2M^2N + M^3))$.

4. Simulation results

We compare the performance of the proposed algorithm with that of the URM [6], PR unitary ESPRIT [15], PR root-MUSIC [16] and PR-URM [17] algorithms in terms of root mean square error (RMSE) performance. For the proposed approach, we always use the DDS [17] to help determine the final DOA estimates. Meanwhile, we examine their ability to remove outliers as well, namely, probability of unsuccessful estimators. The CRLB [20] is plotted as a benchmark. In our simulations, two independent narrowband Gaussian signals are assumed to impinge upon a ULA with $M = 8$ omnidirectional sensors from directions $\theta_1 = 11°$ and $\theta_2 = 18°$. The noise is a zero-mean white Gaussian process. The SNR is defined as the ratio of the power of all source signals to that of the additive noise at each sensor. In all experiments, we assume that the source localization sectors and number of sources are known or estimated by [21–25]. 5000 Monte Carlo simulations are carried out to compute the RMSE, which is defined as

$$\text{RMSE} = \sqrt{\frac{1}{5000P}\sum_{i=1}^{P}\sum_{j=1}^{5000}(\hat{\theta}_{i,j} - \theta_i)^2}. \quad (18)$$

In all experiments, we assume that the number of sources is known. According to (9), we use the conventional beamforming to pre-estimate $\hat{\Theta}$ in each independent run.

4.1. Choices of p and K

In the first example, we study the performance of the proposed method for different values of K, i.e., the number of PR runs. We choose $K = 2$, 5, 10 and 20 for comparison. We vary the SNR from -10 dB to 8 dB, while the number of snapshots is 50. It is seen from Fig. 3 that a larger K provides better performance. When $K = 2$, the proposed scheme has the worst performance. However, if we increase K to 5, the proposed method achieves a visible performance improvement. Compared to $K = 10$, only slight accuracy improvement is attained for $K = 20$. However, the computational complexity of $K = 20$ is twice larger than that of $K = 10$. Thus, K should not be too large or too small. In general, $K = 10$ is an appropriate choice to provide a good performance for our solution.
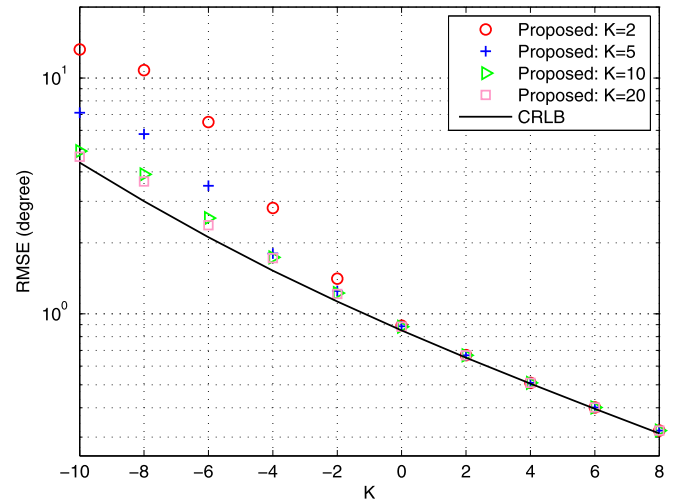

Fig. 3. RMSE versus SNR for different values of K.
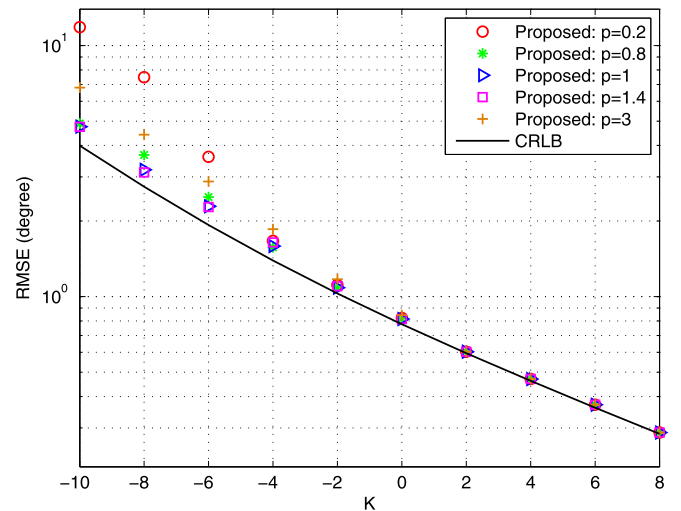

Fig. 4. RMSE versus SNR for different values of p.

We now examine the effects of different pseudo-noise power on the performance of the proposed method. Since the power is controlled by the user defined parameter p, we consider five cases, i.e., $p = 0.2$, $p = 0.8$, $p = 1$, $p = 1.4$ and $p = 3$. We set the number of PR runs as $K = 10$. The number of snapshots is $N = 60$. Fig. 4 shows the RMSE performance versus SNR. It is seen that when p is too small or too large, say, $p = 0.2$ or $p = 3$, the performance of the proposed method is worse than the other three cases. When p approximately equals 1, the proposed method can offer a considerably improved performance, which is not affected by the selection of p. This is because when p is small, the power
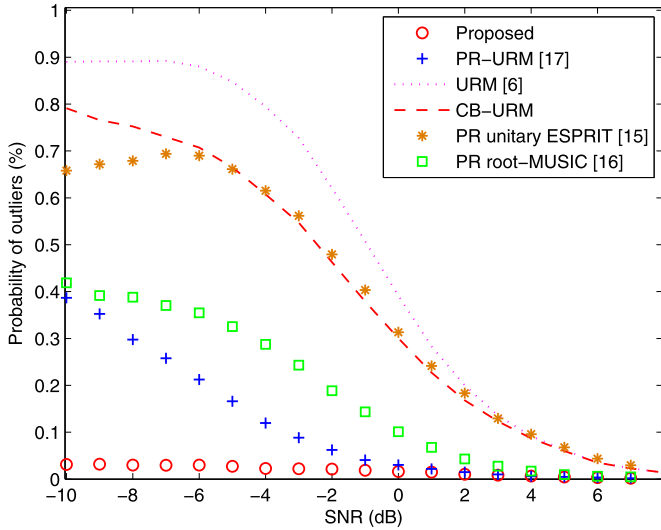
Fig. 5. Probability of unsuccessful estimators versus SNR at $N = 10$.

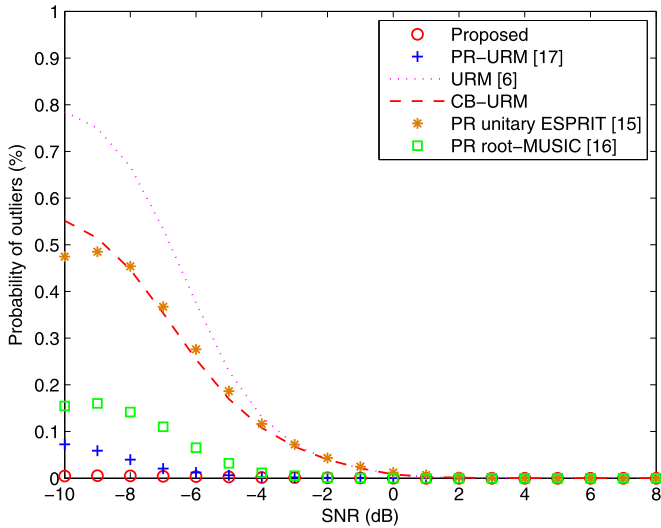**Fig. 7.** RMSE versus SNR. Two-source case: $\theta_1 = 11°$ and $\theta_2 = 18°$.

Fig. 6. Probability of unsuccessful estimators versus SNR at $N = 60$.

of additive pseudo-noise is not large enough to perturb the original noise. Thus, the performance of the proposed method cannot be improved significantly. However, since the pseudo-noise is directly added to the received samples, the modified SNR is lower than the original SNR. Therefore, p should not be too large. In general, $p = 1$ is large enough in most cases. As a result, we set $p = 1$ in the following simulations.

4.2. Performance comparison

In this example, we study the ability of removing outliers under small sample size as a function of SNR. We use probability of unsuccessful estimators which is calculated as the ratio of the number of unsuccessful DOA estimators to the total number of trials to examine the performance of each algorithm in terms of removing outliers. In other words, in each independent run, if there is any DOA estimate localized out of its own DOA sector, we say this estimator is an unsuccessful DOA estimator. We set the number of PR runs as $K = 10$. Figs. 5 and 6 show the probability of unsuccessful estimators for $N = 10$ and $N = 60$, respectively. It is seen that the CB-URM has a better performance than the URM algorithm, and the proposed method provides a considerable performance improvement. When $N = 60$, the proposed algorithm has
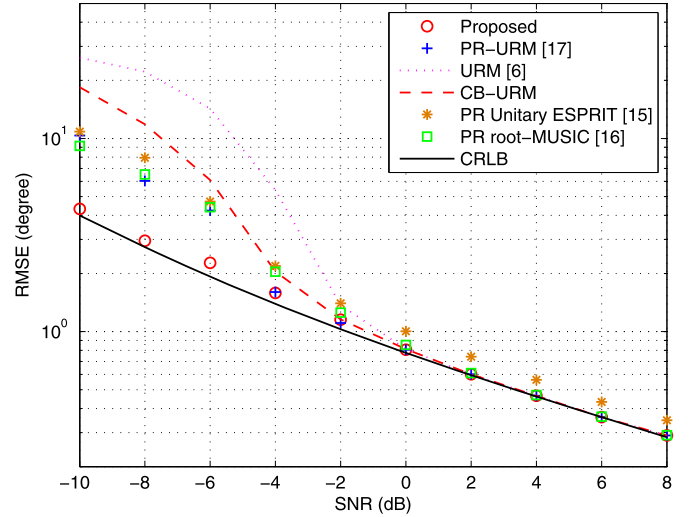
almost removed all the outliers. However, the other DOA estimation algorithms suffer a high probability of unsuccessful estimators no matter whether N is small or large. This also demonstrates that the proposed TSRT has a greater ability to construct a successful DOA estimator from limited number of PR runs compared to the conventional reliability test.

Next, we compare the RMSE performance versus SNR. We vary the signal power such that the input SNR changes from -10 to 8 dB. We set the number of PR runs as $K = 10$ and the number of snapshots is $N = 60$. It is observed from Fig. 7 that the CB-URM method outperforms the URM, and the proposed method achieves the best performance, particularly at small SNRs. However, the performance of other PR based algorithms is somewhat inferior to the proposed approach, because they all need a larger K to obtain the accuracy similar to the proposed one when SNR is low. When the SNR is larger than 2 dB, the proposed and CB-URM algorithms converge. This is due to the fact that, at high SNRs, all the DOA estimates are accepted by the hypothesis test which can also be observed from Fig. 6, and the proposed method is reduced to the CB-URM scheme. To further demonstrate the effectiveness of the proposed method, we consider a severe case when there are three DOAs: $\theta_1 = -5°$, $\theta_2 = 5°$ and $\theta_3 = 30°$. Note that in this case, there are only two peaks of the beamforming output. Since θ_1 and θ_2 are closely spaced, the conventional beamforming cannot resolve them and only one peak is formed for θ_1 and θ_2. The other peak is at θ_3. Thus, the number of DOA sectors is less than the number of sources. Using (9), the estimated $\hat{\Theta}$ is about $\hat{\Theta} = [-11°, 11°] \cup [23°, 38°]$ which contains all the locations of the three DOAs. From Fig. 8, it is seen that the proposed method still works properly and achieves the best performance among all other algorithms.

We now consider a case when two sources are correlated, whereas the number of snapshots is fixed at $N = 40$ and SNR is fixed at -5 dB. In Fig. 9, the RMSE of the estimated DOA is plotted as a function of the correlation coefficient ρ between the two sources. Here, the correlated source samples are generated from a first-order autoregressive process:

$$s_2(i) = \rho s_1(i) + \sqrt{1 - |\rho|^2} \cdot e_1(i), \quad i = 1, \cdots, N \qquad (19)$$

where $e_1(i)$ is independent and identically distributed complex Gaussian noise with zero mean and variance σ_n^2. It can be seen that the performance of the proposed method is almost independent of the correlation, whereas the performance of the PR root-MUSIC deteriorates as ρ increases. Compared to the CB-URM
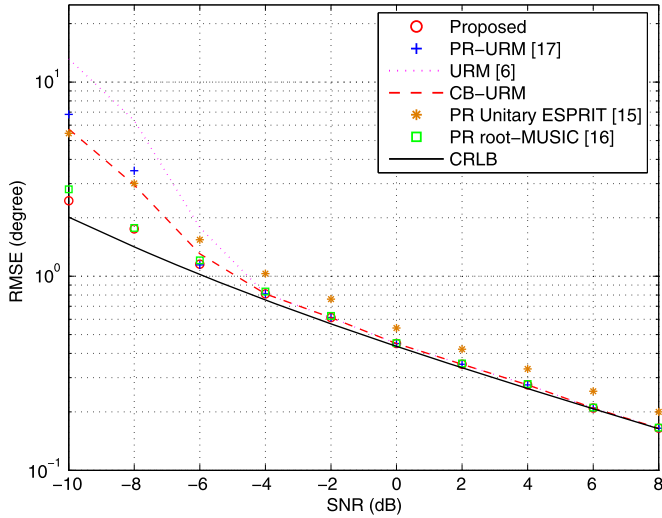
Fig. 8. RMSE versus SNR. Three-source case: $\theta_1 = -5°$, $\theta_2 = 5°$ and $\theta_3 = 30°$.

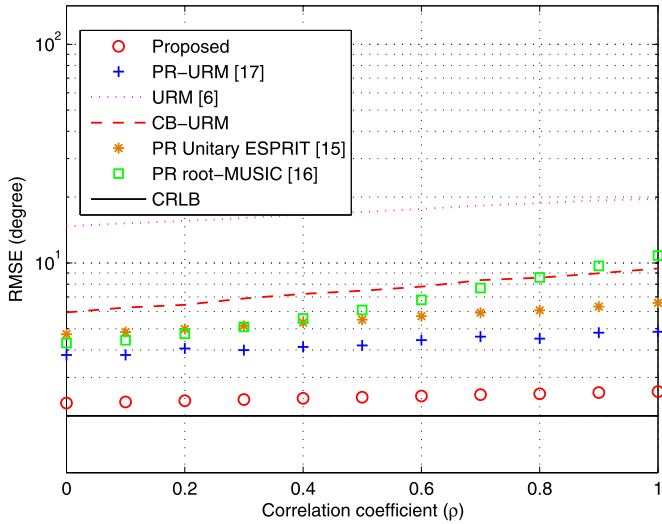**Fig. 9.** RMSE versus correlation coefficient.

Fig. 10. RMSE versus SNR. ESPRIT, unitary ESPRIT, root-MUSIC algorithms, and their respective TSRT and PR based versions are compared.

Fig. 11. RMSE versus number of PR runs.

method, the proposed scheme achieves a considerable angle accuracy improvement.

4.3. Applicability of two-step reliability test

In the final example, we use the proposed TSRT to improve the performance of the ESPRIT [3], unitary ESPRIT [5] and root-MUSIC [19] methods. Here, the root-MUSIC is based on the conventional beamforming and it has been used by the PR root-MUSIC algorithm [16] to form the estimator bank. Hence, we also introduce the PR root-MUSIC for comparison. Since the ESPRIT and unitary ESPRIT algorithms have no signal roots, the DDS [17] cannot be applied. For the purpose of a fair comparison, we use the median to determine the final DOA estimates in all the investigated algorithms. We set the numbers of snapshots and PR runs as $N = 60$ and $K = 10$, respectively. Fig. 10 shows the RMSEs. At small SNRs, the TSRT based ESPRIT, unitary ESPRIT and root-MUSIC show a significant performance improvement compared to their original versions. It is worth noting that the PR unitary ESPRIT algorithm [15] does not provide the performance improvement compared to the unitary ESPRIT approach in the whole SNR region. The PR root-MUSIC scheme [16] outperforms the root-MUSIC, but it is still inferior to the TSRT based one, especially at small SNRs.
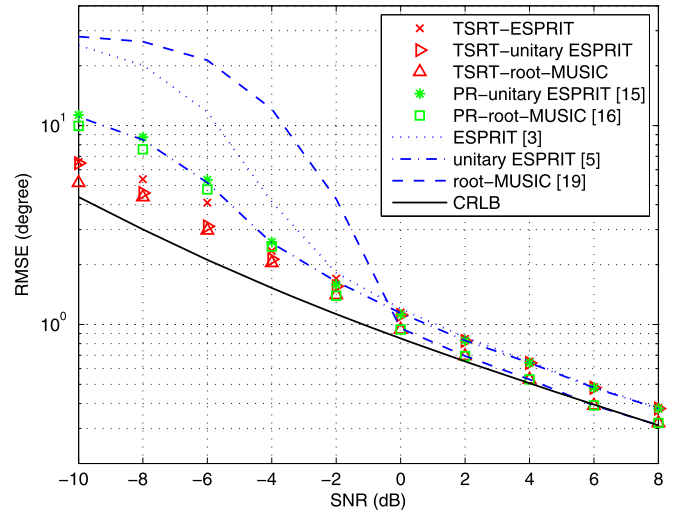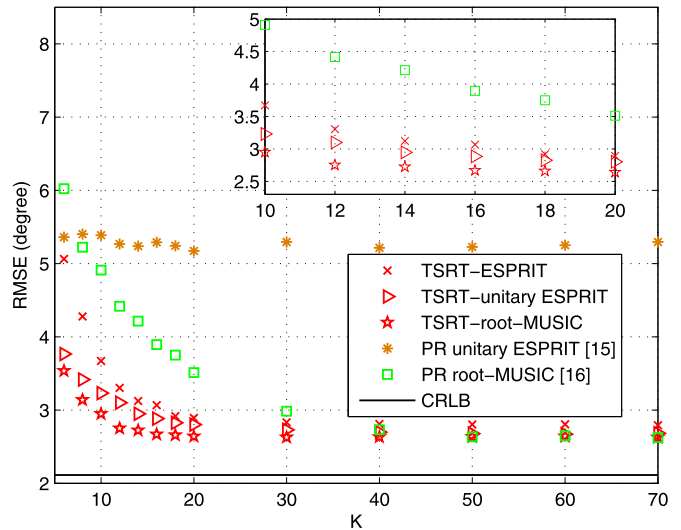
To further demonstrate the advantages of the TSRT, we also study RMSE performance as a function of PR runs. Here, SNR is set to be -6 dB, the number of snapshots is $N = 40$ and K is varied from 1 to 70. The remaining parameters are the same as in Fig. 9. Fig. 10 shows the RMSEs of the TSRT and PR based ESPRIT, unitary ESPRIT and root-MUSIC algorithms as a function of K. Meanwhile, the RMSEs of the PR unitary ESPRIT [15] and PR root-MUSIC [16] are also plotted in Fig. 11 for comparison. It is observed that with K becoming larger, the RMSEs of the TSRT based algorithms gradually decrease. However, the performance of PR unitary ESPRIT [15] is independent of K. When $K \geq 60$, the PR root-MUSIC [16] and the TSRT based schemes merge together. It is worth mentioning that to achieve a comparable RMSE performance of the TSRT at $K = 12$, the number of PR runs that the former needs is about $K = 50$. In this sense, compared to the existing methods, the TSRT needs a much smaller number of PR runs, which reduces the computational burden.

5. Conclusion

A TSRT based URM algorithm for DOA estimation is developed in this paper. The proposed method is able to reduce the number

of PR runs and improve the threshold effect. Unlike the conventional reliability test, we propose a TSRT technique in which the successful DOA estimates of a given DOA estimator can be retained separately and they will not be rejected even if there are DOA estimates rejected by the test. Thus, it is possible to use fewer PR runs to collect sufficient successful DOA estimates that can be used to determine the final DOA estimates. It is worth mentioning that the TSRT can also be applied to other DOA estimation methods. Simulation results verify the effectiveness of the proposed algorithm. As a future work, the proposed algorithm will be extended to DOA estimation for audio signals in microphone array application.

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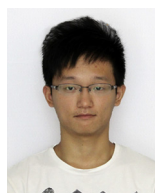
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